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Comparison of Fokker-Planck and Bhatnagar-Gross-Krook Equations

MAHESH S. GREWAL
Aerospace Corporation, Los Angeles, California
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A Fokker-Planck equation, such as the one used in the theory of Brownian motion, and the Bhatnagar-Gross-Krook equation, that conserves only number density, are compared via their transition probabilities. These two equations have been used in various problems arising in the study of plasmas. It is shown that for time intervals large compared with the average collision time, the two equations give the same results. For the opposite limit of times small compared with the collision time, the disparity between the two equations increases as the initial velocity of the test particle increases; however, the difference between the two transition probabilities is shown to be reduced if the initial velocity distribution is Maxwellian.

I. INTRODUCTION

THE Fokker-Planck equation (abbreviated here as the F-P equation) referred to in this article is the one originally introduced in the study of the motion of a Brownian particle. A lucid discussion of it is given in Ref. 1. Another equation under the same name, pertinent to the study of ionized gases, was recently introduced by Rosenbluth, McDonald, and Judd.[2] The connection between these two F-P equations is discussed in Appendix A.

The Bhatnagar-Gross-Krook[3] equation (abbreviated here as the BGK equation) studied in this paper is the one that conserves only the number density; it has also been presented in a form which satisfies all conservation principles. Although originally the BGK equation was derived on a phenomenological basis, some connection between this and the Boltzmann equation can be established.[4]

Because of the mathematical difficulties posed by the use of more appropriate equations, such as the Boltzmann equation or the F-P equation of Rosenbluth *et al.*, recently, several authors[5] have used a Brownian motion F-P equation or BGK equation for the investigation of various problems arising in the study of ionized gases. Both of these latter equations share the following common properties: (a) a given initial velocity distribution will relax to a Maxwellian; (b) the number density is conserved but momentum and energy are not.

Recently the effects of collisions on electron density fluctuations in plasmas have been investigated by Dougherty and Farley[6] using a BGK equation and by the present author[7] using a F-P equation. The remarkable similarity between the results given by these two equations led to the present investigation. For the limiting cases of a large amount of collisions and of no collisions the results were identical. For the collisionless case both equations give the same result for the trivial reason that both reduce to the collisionless Boltzmann equation. However, identical results for the collision-dominated cases were quite unexpected.

In Sec. II the transition probabilities for the F-P equation, for the two limiting cases of time intervals large and small compared with the average collision time, are summarized. In Sec. III corresponding results for the BGK equation are derived and compared with those of the F-P equation. Transition probabilities for both these equations are discussed only for the case of no electric fields. However, for the investigation of problems in plasmas where the BGK or F-P equation is used in the linearized form, e.g., oscillations, fluctuations, ac conductivity, etc., their solution can be constructed from the field-free bivariate transition probability for the corresponding equation (see for example Ref. 7).

II. DISCUSSION OF RESULTS FOR F-P EQUATION

Before proceeding to the analysis of the BGK equation, we summarize the corresponding results for the F-P equation given in Ref. 1. Letting $F(\mathbf{r},t,\mathbf{v})$, denote the joint position and velocity distribution at time t, the F-P equation under consideration may be written as

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}}\right)F(\mathbf{r},t,\mathbf{v})=\beta\frac{\partial}{\partial \mathbf{v}}\cdot\left(\mathbf{v}+\frac{c^2}{2}\frac{\partial}{\partial \mathbf{v}}\right)F(\mathbf{r},t,\mathbf{v}), \quad (1)$$

where β is the effective collision frequency, and $c^2=2\theta/m$ is the average thermal speed in which m and θ denote the mass and temperature, respectively. The transition probability in position only $W(\mathbf{r},t;\,\mathbf{v}_0)$, i.e., the distribution of the displacement $\mathbf{r}$ at time t given that the particle is at $\mathbf{r}=0$ with a velocity $\mathbf{v}_0$ at time $t=0$, can be written as [Eq. (171) of Ref. 1]

$$W(\mathbf{r},t;\,\mathbf{v}_0)=\left\{\frac{\beta^2}{\pi c^2[2\beta t-3+4\exp(-\beta t)-\exp(-2\beta t)]}\right\}^{3/2} \times\exp\left\{-\frac{\beta^2}{c^2}\frac{|\mathbf{r}-\mathbf{v}_0[1-\exp(-\beta t)]/\beta|^2}{[2\beta t-3+4\exp(-\beta t)-\exp(-2\beta t)]}\right\}. \quad (2)$$

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[7] M. S. Grewal, Phys. Rev. **134**, A86 (1964).

The bivariate probability distribution $W(\mathbf{r},\mathbf{v},t;\,\mathbf{v}_0)$ governing the probability of the simultaneous occurrence of the velocity $\mathbf{v}$ and the position $\mathbf{r}$ at time t can also be written down [Eq. (280) of Ref. 1], but since we will evaluate the corresponding expression for the BGK equation only for the case $\beta t \gg 1$, we will present $W(\mathbf{r},\mathbf{v},t;\,\mathbf{v}_0)$ to this approximation at a later point.

For time intervals large compared with β^{-1}, Eq. (2) can be simplified by observing that under these conditions, exponential and constant terms can be neglected in comparison with βt; and that $\langle|\mathbf{r}|^2\rangle_{\text{av}}$ is of the order of c^2t/β, making $\mathbf{v}_0\beta^{-1}$ small in comparison with average $\mathbf{r}$. Thus, for $\beta t\gg 1$, Eq. (2) simplifies to

$$W(\mathbf{r},t;\,\mathbf{v}_0)=(\beta/2\pi c^2t)^{3/2}\exp(-|\mathbf{r}|^2\beta/2c^2t)\,. \tag{3}$$

In the same approximation, the bivariate probability distribution $W(\mathbf{r},\mathbf{v},t;\,\mathbf{v}_0)$ is simply given by the product of $W(\mathbf{r},t;\,\mathbf{v}_0)$ and a Maxwellian velocity distribution in $\mathbf{v}$.

For time intervals small compared with β^{-1} we simplify Eq. (2) by expanding exponentials in powers of βt and retaining the lowest order terms. Thus for $\beta t\ll 1$, Eq. (2) reduces to

$$W(\mathbf{r},t;\,\mathbf{v}_0)=\left(\frac{3}{2\pi\beta c^2t^3}\right)^{3/2}\exp\left(-\frac{3|\mathbf{r}-\mathbf{v}_0t|^2}{2\beta c^2t^3}\right). \tag{4}$$

III. TRANSITION PROBABILITIES FOR BGK EQUATION

Using the nomenclature introduced in Eq. (1), the BGK equation under consideration may be written as

$$\left[\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{r}}\right]F(\mathbf{r},t,\mathbf{v})=-\beta F(\mathbf{r},t,\mathbf{v})+\beta F_0(\mathbf{v})\int F(\mathbf{r},t,\mathbf{v})d\mathbf{v}\,, \tag{5}$$

where $F_0(\mathbf{v})=(\pi c^2)^{-3/2}\exp(-v^2/c^2)$ represents the normalized Maxwellian distribution.

The transition probability for Eq. (5) is analyzed using a Laplace transform with respect to time and a Fourier transform with respect to position. If one defines

$$F(\mathbf{k},s,\mathbf{v})=\int_0^\infty dt\,e^{-st}\int_{-\infty}^{+\infty}\exp(-i\mathbf{k}\cdot\mathbf{r})F(\mathbf{r},t,\mathbf{v})d\mathbf{r}\,,$$

then Eq. (5) reads, on transformation,

$$(s+i\mathbf{k}\cdot\mathbf{v})F(\mathbf{k},s,\mathbf{v})-\delta\{\mathbf{v}-\mathbf{v}_0\}=-\beta F(\mathbf{k},s,\mathbf{v})+\beta F_0(\mathbf{v})\int F(\mathbf{k},s,\mathbf{v})d\mathbf{v}\,, \tag{6}$$

where δ represents Dirac's delta function and we have used $F(\mathbf{r},0,\mathbf{v})=\delta(\mathbf{r})\delta(\mathbf{v}-\mathbf{v}_0)$ for the initial conditions.

It follows from Eq. (6) that

$$F(\mathbf{k},s,\mathbf{v})=\frac{\beta F_0(\mathbf{v})}{s+i\mathbf{k}\cdot\mathbf{v}+\beta}\int F(\mathbf{k},s,\mathbf{v})d\mathbf{v}+\frac{\delta(\mathbf{v}-\mathbf{v}_0)}{s+i\mathbf{k}\cdot\mathbf{v}+\beta}\,. \tag{7}$$

Integrating over $\mathbf{v}$ and denoting $\int F(\mathbf{k},s,\mathbf{v})d\mathbf{v}$ by $N(\mathbf{k},s)$, we obtain, after evaluating the integral $\int[F_0(\mathbf{v})/(s+i\mathbf{k}\cdot\mathbf{v}+\beta)]d\mathbf{v}$,

$$N(\mathbf{k},s)=\frac{1/(s+i\mathbf{k}\cdot\mathbf{v}+\beta)}{1-(\beta\sqrt{\pi}/ck)\exp[(s+\beta)/ck]^2\,\mathrm{erfc}[(s+\beta)/ck]}\,, \tag{8}$$

where $\mathrm{erfc}(x)$ is the complementary error function defined by $\mathrm{erfc}(x)=(2/\sqrt{\pi})\int_x^\infty\exp(-t^2)dt$ (tabulated e.g., in Ref. 8).

Observe that $N(\mathbf{k},s)$ in the transformed representation, has the same interpretation as $W(\mathbf{r},\mathbf{t};\,\mathbf{v}_0)$ introduced earlier in connection with the F-P equation. Different symbols have been introduced to avoid the use of subscripts.

The transformed bivariate probability distribution for the BGK equation can now be obtained from Eq. (7) after substituting $N(\mathbf{k},s)$ from Eq. (8) for $\int F(\mathbf{k},s,\mathbf{v})d\mathbf{v}$ in the right-hand side. To keep the analysis tractable we will invert the transform only for the two limiting cases (a) $\beta t\gg 1$ and (b) $\beta t\ll 1$.

(a) $\beta t\gg 1$

As shall presently be seen from the results given in Eq. (10), c^2k^2 is of the order of βt^{-1} [please note the similarity of this observation with the corresponding one made in the reduction of Eq. (2) to Eq. (3)]. Thus, using the following asymptotic expansion for $\mathrm{erfc}(x)$ for large values of its arguments

$$(\beta/ck)(\sqrt{\pi})\exp\left(\frac{s+\beta}{ck}\right)^2\mathrm{erfc}\left(\frac{s+\beta}{ck}\right)=\frac{\beta}{s+\beta}\times\left\{1-\frac{1}{2}\left(\frac{ck}{s+\beta}\right)^2+\frac{1.3}{4}\left(\frac{ck}{s+\beta}\right)^4+\cdots\right\}.$$

Equation (8) can be written as

$$N(\mathbf{k},s)=\left[(s+\beta)\left\{1-\frac{\beta}{s+\beta}\left[1-\frac{1}{2}\left(\frac{ck}{s+\beta}\right)^2\right]\right\}+i\mathbf{k}\cdot\mathbf{v}_0\left\{1-\frac{\beta}{s+\beta}\left[1-\frac{1}{2}\left(\frac{ck}{s+\beta}\right)^2+\cdots\right]\right\}\right]^{-1}$$
$$=\left[\left\{s+\frac{1}{2}\frac{c^2k^2\beta}{(s+\beta)^2}+\cdots\right\}+i\mathbf{k}\cdot\mathbf{v}_0\left\{1-\frac{\beta}{s+\beta}\right\}\right]^{-1}. \tag{9}$$

[8] B. D. Fried and S. D. Conte, *The Plasma Dispersion Function* (Academic Press Inc., New York, 1961).

For the behavior of $N(\mathbf{k},t)$ for time intervals large compared with β^{-1} we can approximate $\beta/(s+\beta)^2$ by $1/\beta$ and $\beta/(s+\beta)$ by 1, and obtain in the limit of $\beta t\gg 1$ after inverting the Laplace transform

$$N(\mathbf{k},t)=\exp\left(-\frac{k^2c^2t}{2\beta}\right). \tag{10}$$

Inverting the Fourier transform next, Eq. (10) yields

$$N(\mathbf{r},t;\mathbf{v}_0)=[1/(2\pi)^3]\int\exp(-k^2c^2t/2\beta)\exp(i\mathbf{k}\cdot\mathbf{r})d\mathbf{k}$$
$$=(\beta/2\pi c^2t)^{3/2}\exp(-|\mathbf{r}|^2\beta/2c^2t) \tag{11}$$

($\mathbf{v}_0$ is inserted into the arguments of N as an initial condition).

The above expression for the transition probability for the BGK equation is identical with that of the F-P equation presented earlier in Eq. (3). We note that in both cases the distribution of displacement $\mathbf{r}$ is independent of the initial velocity $\mathbf{v}_0$.

Substituting $N(\mathbf{k},s)$ from Eq. (9) for $\int F(\mathbf{k},s,\mathbf{v})d\mathbf{v}$ in the right-hand side of Eq. (7), it reads

$$F(\mathbf{k},s,\mathbf{v})=F_0(\mathbf{v})\left[\left(\frac{s+\beta}{\beta}+\frac{i\mathbf{k}\cdot\mathbf{v}}{\beta}\right)\left(1+\frac{k^2c^2}{2\beta}\right)\right]^{-1}+\frac{\delta(\mathbf{v}_0-\mathbf{v}_0)}{s+i\mathbf{k}\cdot\mathbf{v}+\beta}. \tag{12}$$

Once again, observing the order of magnitude of different terms, i.e., $\mathbf{k}\cdot\mathbf{v}/\beta\sim(\beta t)^{-1/2}$, $(s+\beta)/\beta\sim 1$ etc., we obtain for $\beta t\gg 1$, after effecting the inversion of Laplace and Fourier transforms

$$N(\mathbf{r},t,\mathbf{v};\mathbf{v}_0)=F_0(\mathbf{v})(\beta/2\pi c^2t)^{3/2}\exp(-|\mathbf{r}|^2\beta/2c^2t). \tag{13}$$

It may be observed that for $\beta t\gg 1$, the bivariate probability distribution is also identical for the BGK and the F-P equations.

In passing we remark that, although at the outset it may appear that the transition probabilities for all collision models which relax an initial velocity distribution to a Maxwellian in about a collision time will be roughly the same for $\beta t\gg 1$, this need not be so. We illustrate the point by the following simple example: The transition probability of a BGK equation with a collision term that does not conserve number density but relaxes an initial velocity distribution to a Maxwellian, i.e., with $(\partial f/\partial t)_{\text{coll}}=-\beta f+\beta f_0(\mathbf{v})$, is given by

$$\exp(-\beta t)\delta\{\mathbf{v}-\mathbf{v}_0\}\delta\{\mathbf{r}-\mathbf{v}_0t\}+[1-\exp(-\beta t)]f_0(\mathbf{v}).$$

This expression, for $\beta t\gg 1$ reduces to $f_0(\mathbf{v})$, indicating that not only has the initial velocity distribution become Maxwellian in about one collision time but also the displacement $\mathbf{r}$ has been uniformly distributed over the whole space. This is in contrast to the previous results for the F-P equation and the number-density conserving BGK equation where for $\beta t\gg 1$, although the velocity distribution has become Maxwellian, the distribution of displacement $\mathbf{r}$ is still proceeding in accordance with Eq. (11).

(b) $\beta t\ll 1$

For time intervals small compared with β^{-1}, we rewrite the expression for $N(\mathbf{k},s)$ given in Eq. (8) in a form suitable for expansion in powers of βt,

$$N(\mathbf{k},s)=\frac{\int_0^\infty\exp(-i\mathbf{k}\cdot\mathbf{v}_0t)\exp(-\beta t)\exp(-st)dt}{1-\beta\int_0^\infty\exp(-k^2c^2t^2/4)\exp(-\beta t)\exp(-st)dt}. \tag{14}$$

On expanding $\exp(-\beta t)$ in powers of βt, Eq. (14) reads

$$N(\mathbf{k},s)=\frac{\int_0^\infty\exp(-i\mathbf{k}\cdot\mathbf{v}_0t)\exp(-st)dt-\beta\int_0^\infty t\exp(-i\mathbf{k}\cdot\mathbf{v}_0t)\exp(-st)dt+\cdots}{1-\beta\int_0^\infty\exp(-k^2c^2t^2/4)\exp(-st)dt+\beta^2\int_0^\infty t\exp(-k^2c^2t^2/4)\exp(-st)dt+\cdots}. \tag{15}$$

After carrying out the division and collecting the terms up to first order in βt, it follows from Eq. (15) that

$$N(\mathbf{k},s)=\int_0^\infty\exp(-i\mathbf{k}\cdot\mathbf{v}_0t)\exp(-st)dt-\beta\left[\int_0^t t\exp(-i\mathbf{k}\cdot\mathbf{v}_0t)\exp(-st)dt-\int_0^\infty\exp(-i\mathbf{k}\cdot\mathbf{v}_0t)\exp(-st)dt\int_0^\infty\exp(-k^2c^2t^2/4)\exp(-st)dt\right]. \tag{16}$$

After inverting the Laplace transform, the preceding equation yields

$$N(\mathbf{k},t)=\exp(-i\mathbf{k}\cdot\mathbf{v}_0t)(1-\beta t)+\int_0^t \exp[-i\mathbf{k}\cdot\mathbf{v}_0(t-\tau)]\exp(-k^2c^2\tau^2/4)d\tau. \tag{17}$$

The Laplace transform inversion of the last term on right-hand side of Eq. (16) is performed using the convolution theorem.

Next, inverting the Fourier transform, we finally obtain for the transition probability of the BGK equation for $\beta t\ll 1$,

$$N(\mathbf{r},t;\,\mathbf{v}_0)=\delta(\mathbf{r}-\mathbf{v}_0t)(1-\beta t)+\frac{\beta}{\pi^{3/2}c^2}\int_0^t\frac{1}{\tau^3}\exp\left\{-\frac{|\mathbf{r}_0-\mathbf{v}_0(t-\tau)|^2}{c^2\tau^2}\right\}d\tau. \tag{18}$$

Before comparing $N(\mathbf{r},t;\,\mathbf{v}_0)$ given in Eq. (18) with the corresponding expression for the F-P equation, we present another derivation of the results given in Eq. (18) based on a physical interpretation of the BGK equation. For this purpose, Eq. (5) is rewritten as

$$\left[\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{r}}\right]F(\mathbf{r},t,\mathbf{v})=-\beta F(\mathbf{r},t,\mathbf{v})+\beta F_0(\mathbf{v})\int F(\mathbf{r},t,\mathbf{v})d\mathbf{v}. \tag{5}$$

The operator appearing on the right-hand side of this equation represents the change in the distribution function due to collisions. Thus, if it were not for the left-hand side the distribution function would remain constant along the characteristics, $\mathbf{r}-\mathbf{v}t$. The two terms appearing on the right-hand side, which represent the effects of collisions may be interpreted as follows: The particles in a range $d\mathbf{v}$ about velocity $\mathbf{v}$ are, according to the first term, absorbed at a rate proportional to $F(\mathbf{r},t,\mathbf{v})$ at $(\mathbf{r},t)$ and by virtue of the second term are re-emitted at the same instant with a Maxwellian distribution $F_0(\mathbf{v})$. Thus if we ignore for a moment the re-emitted particles, then the transition probability of a particle which started at $\mathbf{r}=0$ at $t=0$ with velocity $\mathbf{v}_0$ is given by

$$N_0(\mathbf{r},t;\,\mathbf{v}_0)=\delta(\mathbf{r}-\mathbf{v}_0t)\exp(-\beta t),$$

which to first order in βt is

$$N_0(\mathbf{r},t;\,\mathbf{v}_0)=\delta(\mathbf{r}-\mathbf{v}_0t)(1-\beta t). \tag{19}$$

Next we trace the evolution of the re-emitted particles. Up to the order of βt to which we are working, the fraction of the original particle absorbed and also re-emitted, at say $t=\tau$ in a range $\delta\tau$ at position $\mathbf{r}(\tau)=\mathbf{v}_0\tau$, along the straight line trajectory of the original particle, is equal to $\beta\delta\tau$. Once again Eq. (5) governs the transport of this fraction. At the instant of emission, because of its Maxwellian velocity distribution, the collision terms of Eq. (5) go to zero, and the fraction $\beta\delta\tau$ spreads according to the left-hand side. But the moment this fraction leaves the source point $\mathbf{r}(\tau)$ the distribution is no longer Maxwellian at any $\mathbf{r}$, and the collision terms come into play again. However, as seen earlier, the effects of the collisions give rise to terms of the order of βt and since the starting density $\beta\delta\tau$ is itself of this order, the resulting terms are of the second order and may therefore be neglected. Thus for the re-emitted particles we can neglect the right-hand side of Eq. (5). Governed only by the operator appearing on the left-hand side, the distribution of the fraction $\beta\delta\tau$ emitted at time τ after a time interval $t-\tau$ is given by

$$N_\tau(\mathbf{r},\,t-\tau)=(\beta\delta\tau/\Pi^{3/2}c^3)\int\delta\{\mathbf{r}-[\mathbf{r}(\tau)+\mathbf{v}\cdot(t-\tau)]\}\times\exp(-v^2/c^2)d\mathbf{v},$$

which after evaluating the integral yields

$$N_\tau(\mathbf{r},\,t-\tau)=\frac{\beta\delta\tau}{\Pi^{3/2}c^3}\frac{1}{(t-\tau)^3}\exp\left\{-\frac{|\mathbf{r}-\mathbf{r}(\tau)|^2}{c^2(t-\tau)^2}\right\}. \tag{20}$$

To get the complete distribution of the displacement $\mathbf{r}$ at time t of the particle that started at $\mathbf{r}=0$ at $t=0$ with velocity $\mathbf{v}_0$, we integrate Eq. (20) over τ from $\tau=0$ to $\tau=t$, with $\mathbf{r}(\tau)=\mathbf{v}_0\tau$, and add to the result of Eq. (19) to obtain

$$N(\mathbf{r},t;\,\mathbf{v}_0)=\delta(\mathbf{r}-\mathbf{v}_0t)(1-\beta t)+\frac{\beta}{\Pi^{3/2}c^3}\int_0^t(t-\tau)^{-3}\times\exp\left\{-\frac{|\mathbf{r}-\mathbf{v}_0\tau|^2}{c^2(t-\tau)^2}\right\}d\tau. \tag{21}$$

Equation (21) becomes identical with Eq. (18) after changing the variable of integration from τ to $t-\tau$.

Returning to Eq. (18), we introduce the following nondimensional parameters:

$$\nu=|v_0|/c;\quad \tau=\beta t;\quad \boldsymbol{\xi}=(\beta/c)(\mathbf{r}-\mathbf{v}_0t);$$
$$\boldsymbol{\zeta}=\boldsymbol{\xi}\cdot\mathbf{v}_0/v_0;\quad \boldsymbol{\eta}=\boldsymbol{\xi}-\boldsymbol{\zeta}. \tag{22}$$

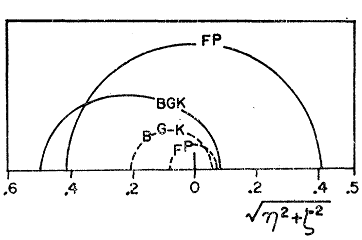

FIG. 1. Cross sections of surfaces of probability distribution $N=W=1.0$ for F-P, and BGK equations for $\nu=1.0$. Dashed curves are for $\beta t=0.1$ and solid ones for $\beta t=0.5$.

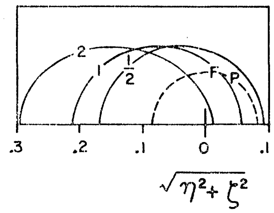

FIG. 2. Cross sections of surfaces of probability distribution $N=W=1.0$ for F-P, and BGK equations for $\beta t=0.1$. The solid curves are for BGK equation for the different values of ν indicated.

In terms of these dimensionless quantities, and after evaluating the integral, Eq. (18) reads

$$N(\xi,\tau;\nu)=(\beta^3/c^3)\delta(\xi)(1-\tau)+\frac{\beta^3}{8(\sqrt{\pi})c^3}\frac{1}{\eta^2+\zeta^2}\operatorname{der}\left[\frac{\nu\eta}{(\eta^2+\zeta^2)^{1/2}}\right]\times\left\{\operatorname{der}\left[\frac{(\eta^2+\zeta^2)^{1/2}}{\tau}+\frac{\nu\zeta}{(\eta^2+\zeta^2)^{1/2}}\right]-\frac{2\nu\zeta}{(\eta^2+\zeta^2)^{1/2}}\times\operatorname{erfc}\left[\frac{(\eta^2+\zeta^2)^{1/2}}{\tau}+\frac{\nu\zeta}{(\eta^2+\zeta^2)^{1/2}}\right]\right\}, \quad (23)$$

where $\operatorname{erfc}(x)$ has been defined earlier following Eq. (8) and $\operatorname{der}(x)$ represents the derivative of the error function defined by $\operatorname{der}(x)=(2/\sqrt{\pi})\exp(-x^2)$.

In terms of the same dimensionless quantities the corresponding expression for the F-P equation, given earlier in Eq. (4) can be written as

$$W(\xi,\tau;\nu)=(\beta^3/c^3)(3/2\pi\tau^3)^{3/2}\exp(-3\xi^2/2\tau^3). \quad (24)$$

We observe from Eq. (23) that the transition probability for the BGK equation, unlike that of the F-P equation, is neither spherically symmetric nor independent of the initial velocity in the coordinate system moving with velocity $\mathbf{v}_0$, which has been tacitly introduced through the set of Eqs. (22). This lack of symmetry and added dependence on the parameter ν of Eq. (23) makes it difficult to compare with Eq. (24). The two expressions given in Eqs. (23) and (24) are compared by considering surfaces of equal probability distribution. Figures 1 and 2 show the cross sections of such surfaces for which $N=W=1.0$. From Fig. 1, which shows the evolution of this surface in time, we see that for small time ($\tau\sim0.1$) the surface corresponding to the BGK equation has advanced further than the corresponding one for the F-P equation; for comparatively large times ($\tau\sim0.5$) it is the other way around. Figure 2 shows the surface, once again for $N=W=1$, drawn for different values of ν. It is seen that the disparity in the behavior of the two equations increases as ν increases.

So far we have compared the transition probabilities for the two equations when a definite initial velocity $\mathbf{v}_0$ is prescribed. Now we ask for the distribution of displacement $\mathbf{r}$ according to the two equations when the initial velocity has a Maxwellian distribution.

Averaging Eq. (18) over a Maxwellian distribution in the initial velocity $\mathbf{v}_0$, we find after some algebra

$$N(\mathbf{r},t)=\frac{\exp(-r^2/c^2t^2)}{\pi^{3/2}c^3t^3}\left[1-\beta t+\frac{2\beta t}{(r/ct)}\times\exp(-r^2/c^2t^2)\int_0^{r/ct}\exp(x^2)dx\right]. \quad (25)$$

[The function $\exp(-y^2)\int_0^y\exp(x^2)dx$ has been tabulated; see, for example, Ref. 8.]

The corresponding expression for the F-P equation obtained by averaging Eq. (4) over a Maxwellian distribution is

$$W(\mathbf{r},t)=\frac{1}{\pi^{3/2}c^3t^3}(1+2\beta t/3)^{-3/2}\times\exp\left[-(r^2/c^2t^2)\frac{1}{1+2\beta t/3}\right]. \quad (26)$$

Figure 3 shows the probability of finding the particle, according to Eqs. (25) and (26), in a sphere of thickness $d(r/ct)$ as a function of its radius r/ct for βt equal to 0.1 and 0.5. As can be readily seen, the difference between the distribution of the displacement r as given by the F-P equation and the BGK equation is greatly reduced for $\beta t=0.1$ when the initial velocity is prescribed by a Maxwellian distribution. However, as seen from Fig. 3, this difference increases with βt. Since the two equations give identical results for $\beta t\gg1$, it seems probable that maximum disparity between the two equations occurs around the βt value of 1.

From the proceeding analysis on the comparison of the F-P and BGK equations we see that for the investigation of problems in plasmas where one is interested in time intervals large compared with the average collision time β^{-1}, the two equations will yield the same results. When the time intervals of interest are small compared with β^{-1}, the two equations will again give comparable results in those problems where thermodynamic equilibrium is assumed, and thus the initial velocity of the particle is taken as Maxwellian. Such would be the case, for example, in the study of density fluctuations in a plasma at thermodynamic equilibrium. The disparity

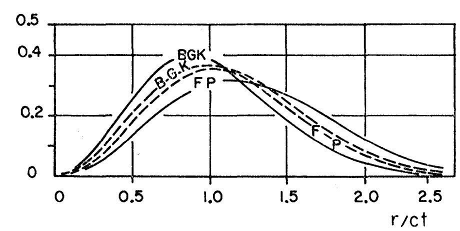

FIG. 3. Distribution of displacement r, for BGK and F-P equations, when the initial velocity is taken as Maxwellian. Dashed curves are for the value of $\beta t=0.1$ and solid ones for $\beta t=0.5$.

between the results given by these two equations will probably be the greatest when the problem under consideration involves study of the dispersion of particles by collisions with an anisotropic initial velocity distribution, in time intervals close to a collision time β^{-1}. Such a problem arises in the study of the ac conductivity of a plasma for frequencies close to β.

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APPENDIX A

In this appendix it is shown that by making approximations relevant to the motion of a Brownian particle, the collision term of the F-P equation of Rosenbluth *et al.* can be reduced to that of the Brownian F-P equation. The collision term of the F-P equation with the use of a superscript b for quantities pertaining to the Brownian particle, may be written as, e.g., see Ref. 9,

$$C=-\nabla_{\mathbf{v}}\langle\Delta v\rangle f^b(\mathbf{v})+\tfrac{1}{2}\nabla_{\mathbf{v}}\nabla_{\mathbf{v}}:\{\langle\Delta v\Delta v\rangle f^b(\mathbf{v})\}\,, \quad \text{(A1)}$$

where $\nabla_{\mathbf{v}}$ means gradient in velocity space and, $\langle\Delta v\rangle$ and $\langle\Delta v\Delta v\rangle$, which are functions of $\mathbf{v}$, are given by

$$\begin{aligned}\langle\Delta v\rangle&=\Gamma\sum_a(1+m^b/m^a)\nabla_{\mathbf{v}}\int d\mathbf{v}'f^a(\mathbf{v}')\frac{1}{|\mathbf{v}-\mathbf{v}'|}\,,\\ \langle\Delta v\Delta v\rangle&=\Gamma\nabla_{\mathbf{v}}\nabla_{\mathbf{v}}\sum_a\int d\mathbf{v}'f^a(\mathbf{v}')\,|\mathbf{v}-\mathbf{v}'|\,,\end{aligned} \quad \text{(A2)}$$

in which summation is to be carried out over all the species present including the one whose change in distribution function is evaluated. The constant Γ is defined in Ref. 9.

For the motion of a Brownian particle, where we are concerned with the change in the distribution function of a small number of particles of colloidal size immersed in a fluid, we make the following observations:

(1) The number density of the Brownian particles is very much smaller than that of the background fluid; thus the collisions that a Brownian particle suffers are almost entirely with the particles of background fluid. As a result, the summations in Eq. (A2) include terms only pertaining to the fluid, which we later on denote by superscript f;

(2) the mass of the Brownian particle is much larger than that of a fluid particle; thus $(1+m^b/m^f)\sim m^b/m^f$;

(3) the velocity of the fluid particles is Maxwellian.

Using these approximations after evaluating the integrals, we obtain from Eq. (A2)

$$\begin{aligned}\langle\Delta v\rangle&=\Gamma n^f(m^b/m^f)\nabla_{\mathbf{v}}[(1/\mathbf{v})\,\mathrm{erf}(\mathbf{v}/c^f)]\,,\\ \langle\Delta v\Delta v\rangle&=\Gamma n^f(1/\mathbf{v})\,\mathrm{erf}(\mathbf{v}/c^f)\,,\end{aligned} \quad \text{(A3)}$$

where c^f is the average thermal speed of the fluid particles. Now the temperatures of the Brownian particles and the fluid are of the same order. Since the Brownian particle is much heavier than a fluid particle, v/c^f is, for most of $f^b(v)$, a very small number; thus expanding the error function for small values of its argument, we obtain from Eq. (A3)

$$\begin{aligned}\langle\Delta v\rangle&=-\Gamma n^f(m^b/m^f)(2/\sqrt{\pi})^3\mathbf{v}/(c^f)^3\,,\\ \langle\Delta v\Delta v\rangle&=\Gamma n^f(2/\sqrt{\pi})^3 1/c^f\,.\end{aligned} \quad \text{(A4)}$$

If we now let β denote $\Gamma n^f(m^b/m^f)(2/\sqrt{\pi})^3(c^f)^{-3}$ and use the fact that $(c^f)^2=2\theta^f/m^f$ and $\theta^f\sim\theta^b$, we find

$$\begin{aligned}\langle\Delta v\rangle&=-\beta\mathbf{v}\,,\\ \tfrac{1}{2}\langle\Delta v\Delta v\rangle&=\beta\theta^b/m^b\,.\end{aligned} \quad \text{(A5)}$$

The expressions from Eq. (A5), when substituted in Eq. (A1), yield the collision terms for the Brownian motion F-P equation.

Finally we remark that since the F-P equation of Rosenbluth *et al.* may be looked upon as a simplification of the Boltzmann equation valid for small-angle collisions, the preceding analysis shows a relation between the Boltzmann equation and the Brownian-motion F-P equation. In the derivation of the F-P equation of Rosenbluth *et al.* from the Boltzmann equation, the retaining of only small-angle collision terms was based on the observation that for plasmas, where long-range Coulomb forces govern the interaction of particles, the accumulative effect of the frequent small-angle collisions outweighs the effect of the comparatively infrequent large-angle collisions. In the case of the motion of a Brownian particle, the neglect of large-angle collisions can be justified by the fact that the mass of a Brownian particle is much larger than that of the fluid particle with which it collides.

[9] A. N. Kaufman, *The Theory of Neutral and Ionized Gases*, edited by C. DeWitt and J. F. Detouf (John Wiley & Sons, Inc., New York, 1960).